QUADRATIC EQUATIONS

1. QUADRATIC EQUATION

An equation $ax^2 + bx + c = 0$ (where $a \neq 0$, and $a,b,c \in C$), is called a quadratic equation. Here a, b and c are called coefficients of the equation. This equation always has two roots. Let the roots be α and β.

Then $\alpha + \beta = -\frac{b}{a}$ (sum of the roots) $\alpha\beta = \frac{c}{a}$ (product of the roots)

The quantity $D = b^2 - 4ac$ is called discriminant of the equation.

Roots of the equation are given by $x = \frac{-b \pm \sqrt{D}}{2a}$

2. NATURE OF ROOTS

(i) If $D > 0$, roots of the equation are real and distinct.

(ii) If $D = 0$, roots of the equation are real and equal.

(iii) If $D < 0$, roots of the equation are imaginary.

(iv) If D is square of a rational number, roots of the equation are rational. (Provided a, b and c are rational numbers)

(v) If D >0 and D is not a perfect square, then roots are irrational

3. FORMATION OF AN EQUATION WITH GIVEN ROOTS

A quadratic equation whose roots are α and β is given by

$(x - \alpha)(x - \beta) = 0$

$\Rightarrow x^2 - \alpha x - \beta x + \alpha\beta = 0$

$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$

i.e., x^2 – (Sum of roots)x + Product of roots = 0

TRANSFORMATION OF AN EQAUTION

If α, β are roots of the equation $ax^2 + bx + c = 0$, then the equation whose roots are

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$ is $cx^2 + bx + a = 0$ (Replace x by $\frac{1}{x}$)

(ii) $-\alpha, -\beta$ is $ax^2 - bx + c = 0$ (Replace x by $-x$)

(iii) $k + \alpha, k + \beta$ is $a(x-k)^2 + b(x-k) + c = 0$ {Replace x by $(x-k)$}

(iv) $\alpha^n, \beta^n \ (n \in N)$ is $a(x^{1/n})^2 + b(x^{1/n}) + c = 0$ (Replace x by $x^{1/n}$)

(v) $\alpha^{1/n}, \beta^{1/n} (n \in N)$ is $a(x^n)^2 + b(x^n) + c = 0$ (Replace x by x^n)

(vi) $k\alpha, k\beta$ is $ax^2 + kbx + k^2c = 0$ (Replace x by $\frac{x}{k}$)

(vii) $\frac{\alpha}{k}, \frac{\beta}{k}$ is $k^2ax^2 + kbx + c = 0$ (Replace x by kx)

5. ROOTS IN SPECIAL CASES

$ax^2 + bx + c \equiv a(x-\alpha)(x-\beta)$ (If α and β are roots of the equation)

If $b^2 - 4ac > 0$	Then	
$a > 0, b > 0, c > 0$	$\alpha + \beta < 0, \alpha\beta > 0$	Both roots are negative
$a > 0, b > 0, c < 0$	$\alpha + \beta < 0, \alpha\beta < 0$	Roots are opposite in sign. Magnitude of negative root is more than the magnitude of positive root.
$a > 0, b < 0, c > 0$	$\alpha + \beta > 0, \alpha\beta > 0$	Both roots are positive
$a > 0, b < 0, c < 0$	$\alpha + \beta > 0, \alpha\beta < 0$	Roots are opposite in sign.Magnitude of positive root is more than the magnitude of negative root.

6. CONDITION FOR COMMON ROOT(S)

(i) Let $ax^2 + bx + c = 0$ and $dx^2 + ex + f = 0$ have a common root α (say). Then $a\alpha^2 + b\alpha + c = 0$ and $d\alpha^2 + e\alpha + f = 0$. Then

$$\Rightarrow \quad (dc - af)^2 = (bf - ce)(ae - bd)$$

which is the required condition for the two equations to have a common root.

(ii) Condition for both the roots to be common is $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

8. GREATEST AND LEAST VALUES OF QUADRATIC EXPRESSION

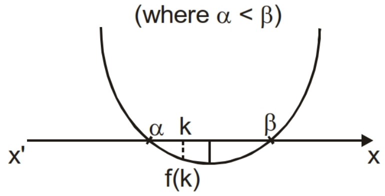
(i) If $a > 0$, then the quadratic expression $ax^2 + bx + c$ has least value $\frac{4ac - b^2}{4a}$ at $x = -\frac{b}{2a}$

(ii) If $a < 0$, then the quadratic expression $ax^2 + bx + c$ has greatest value $\frac{4ac - b^2}{4a}$ at $x = -\frac{b}{2a}$

10. LOCATION OF ROOTS (INTERVAL IN WHICH ROOTS LIE)

In some problems we want the roots α and β of the equation $ax^2 + bx + c = 0$ to lie in a given interval. For this we impose conditions on a, b and c. Since $a \neq 0$, we can take $f(x) = ax^2 + bx + c$.

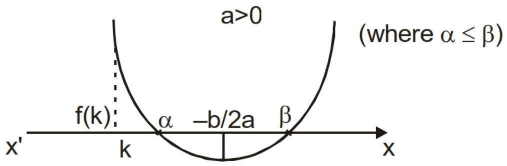
(i) A given number k will lie between the roots if $f(k) < 0$, $D > 0$.



In particular, the roots of the equation will be of opposite signs if 0 lies between the roots $\Rightarrow f(0)<0$.

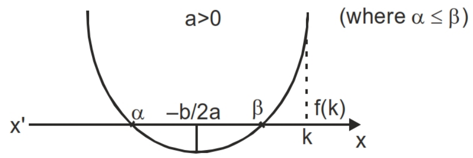
(ii) Both the roots are greater than given number k if the following three conditions are satisfied

$$D \geq 0, \quad -\frac{b}{2a} > k \ \text{ and } \ f(k) > 0$$



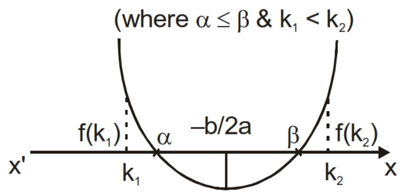
(iii) Both the roots will be less than a given number k if the following conditions are satisfied:

$$D \geq 0, \quad -\frac{b}{2a} < k \ \text{ and } \ f(k) > 0$$

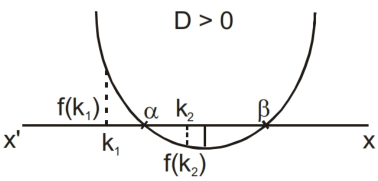


(iv) Both the roots will lie in the given interval (k_1, k_2) if the following conditions are satisfied:

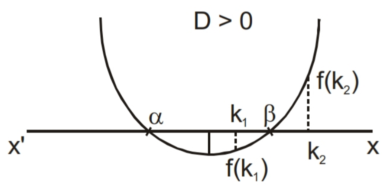
$$D \geq 0, \quad k_1 < -\frac{b}{2a} < k_2 \ \text{ and } \ f(k_1) > 0, \ \ f(k_2) > 0$$



(v) Exactly one of the root lies in the given interval (k_1, k_2) if $f(k_1).f(k_2) < 0$



$f(k_1) > 0$ & $f(k_2) < 0$



$f(k_1) < 0$ & $f(k_2) > 0$

11. THEORY OF EQUATIONS

(i) If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, $(a_0, \ldots\ldots, a_n \in R$ and $a_n \neq 0)$ then $p(x) = 0$ has n roots. (real /complex)

(ii) A polynomial equation in x of odd degree has at least one real root (it has odd no. of real roots).

(iii) If $x_1, \ldots, x_n$ are the roots of $p(x) = 0$, then p(x) can be written in the form $p(x) \equiv a_n(x - x_1)\ldots(x - x_n)$.

(iv) If α is a root of $p(x) = 0$, then $(x - \alpha)$ is a factor of p(x) and vice - versa.

(v) If $x_1,\ldots,x_n$ are the roots of $p(x) \equiv a_n x^n + \ldots + a_0 = 0$, $a_n = 0$.

Then $\sum_{i=1}^{n} x_i = -\frac{a_{n-1}}{a_n}$, $\sum_{1\le i<j\le n} x_i x_j = \frac{a_{n-2}}{a_n}$, $x_1 x_2 \ldots x_n = (-1)^n \frac{a_o}{a_n}$.

(vi) If equation $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$ has more than n distinct roots then f(x) is identically zero.

(vii) If p(a) and p(b) (a < b) are of opposite signs, then p(x) = 0 has odd number of real roots in (a, b), i.e. it has at least one real root in (a, b) and if p(a) and p(b) are of same sign then p(x) = 0 has even number of real roots in (a, b).

(ix) If coefficients of p(x) (polynomial in x written in descending order) have 'm' changes in signs, then polynomial equation of n^{th} degree p(x) = 0 have at the most 'm' positive real roots and if p(– x) have 't' changes in sign, then p(x) = 0 have at most 't' negative real roots. By this we can find maximum number of real roots.

No. of complex roots = n – (m + t) where m + t < n.

(x) Imaginary roots of a quadratic equation always occur in conjugate pair.

(xi) Irrational roots of a quadratic equation always occur in conjugate pair.

12. QUADRATIC EXPRESSION IN TWO VARIABLE

The general form of a quadratic expression in two variables x, y is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

The condition that this expression may be resolved into two linear rational factors is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

or

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(i) If roots of quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ are in the same ratio $\left(\text{i.e. } \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}\right)$ then $\frac{b_1^2}{b_2^2} = \frac{a_1c_1}{a_2c_2}$

(ii) If one root is k times the other root of quadratic equation $a_1x^2 + b_1x + c_1 = 0$ then $\frac{(k+1)^2}{k} = \frac{b^2}{ac}$